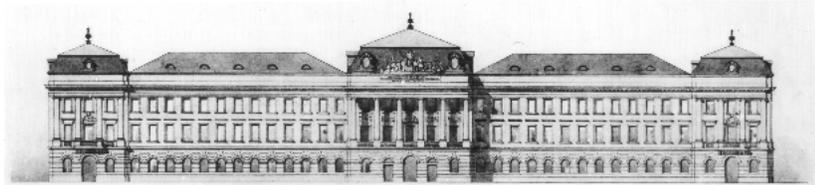


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**INSTITUT FÜR INFORMATIONSSYSTEME
ABTEILUNG WISSENSBASIERTE SYSTEME**

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ON GRAPH CONTRACTIONS AND INDUCED MINORS

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Abstract

The INDUCED MINOR CONTAINMENT problem takes as input two graphs G and H , and asks whether G has H as an induced minor. We show that this problem is fixed parameter tractable in $|V_H|$ if G belongs to any nontrivial minor-closed graph class and H is a planar graph. For a fixed graph H , the H -CONTRACTIBILITY problem is to decide whether a graph can be contracted to H . The computational complexity classification of this problem is still open. So far, H has a dominating vertex in all cases known to be solvable in polynomial time, whereas H does not have such a vertex in all cases known to be NP-complete. Here, we present a class of graphs H with a dominating vertex for which H -CONTRACTIBILITY is NP-complete. We also present a new class of graphs H for which H -CONTRACTIBILITY can be solved in polynomial time. Finally, we study the (H, v) -CONTRACTIBILITY problem, where v is a vertex of H . The input of this problem is a graph G and an integer k , and the question is whether G is H -contractible such that the “bag” of G corresponding to v contains at least k vertices. We show that this problem is NP-complete whenever H is connected and v is not a dominating vertex of H .

1 Introduction

There are several natural and elementary algorithmic problems that check if the structure of some fixed graph H shows up as a *pattern* within the structure of some input graph G . This paper studies the computational complexity of two such problems, namely the problems of deciding if a graph G can be transformed into a graph H by performing a sequence of edge contractions and vertex deletions, or by performing a sequence of edge contractions only. Theoretical motivation for this research can be found in several papers [3, 8, 14, 15] and comes from hamiltonian graph theory [12] and graph minor theory [17], as we will explain below. Practical applications include surface simplification in computer graphics [1, 4] and cluster analysis of large data sets [5, 11, 13]. In the first practical application, graphic objects are represented using (triangulated) graphs and these graphs need to be simplified. One of the techniques to do this is by using edge contractions. In the second application, graphs are coarsened by means of edge contractions.

Basic Terminology. All graphs in this paper are undirected, finite, and have neither loops nor multiple edges. For a graph G and a set of vertices $S \subseteq V_G$, we write $G[U]$ to denote the subgraph of G induced by U . Two sets $S, S' \subseteq V_G$ are called *adjacent* if there exist vertices $s \in S$ and $s' \in S'$ such that $ss' \in E_G$. Let G and H be two graphs. The *edge contraction* of edge $e = uv$ in G removes u and v from G , and replaces them by a new vertex adjacent to precisely those vertices to which u or v were adjacent. If H can be obtained from G by a sequence of edge contractions, vertex deletions and edge deletions, then G contains H as a *minor*. If H can be obtained from G by a sequence of edge contractions and vertex deletions, then G contains H as an *induced minor*. If H can be obtained from G by a sequence of edge contractions, then G is said to be *contractible to H* and G is called *H -contractible*. This is equivalent to saying that G has a so-called *H -witness structure* \mathcal{W} , which is a partition of V_G into $|V_H|$ sets $W(h)$, called *H -witness sets*, such that each $W(h)$ induces a connected subgraph of G and for every two $h_i, h_j \in V_H$, witness sets $W(h_i)$ and $W(h_j)$ are adjacent in G if and only if h_i and h_j are adjacent in H . Here, two subsets A, B of V_G are called *adjacent* if there is an edge $ab \in E_G$ with $a \in A$ and $b \in B$. By contracting all the edges in each of the witness sets, we obtain the graph H . See Figure 1 for an example that shows that in general the witness sets $W(h)$ are not uniquely defined.

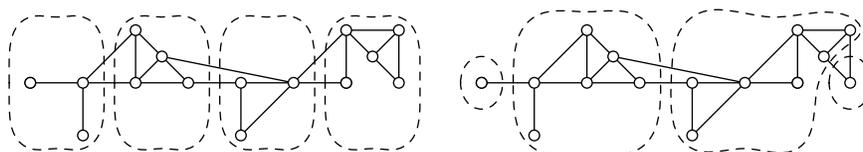


Figure 1: Two P_4 -witness structures of a graph.

For any fixed graph H , the problems H -MINOR CONTAINMENT, H -INDUCED MINOR CONTAINMENT and H -CONTRACTIBILITY ask if an input graph G has H as a minor, has H as an induced minor, or is H -contractible, respectively. When H is part of the input, we denote the three problems by MINOR CONTAINMENT, INDUCED MINOR CONTAINMENT and CONTRACTIBILITY.

Known Results. A celebrated result by Robertson and Seymour [17] states that H -MINOR CONTAINMENT can be solved in cubic time for every fixed graph H . The complexity classification of the other two problems is still open, although Matoušek and Thomas [16] showed that when H is part of the input both problems are already NP-complete when H and G are trees of bounded diameter or trees in which all vertices, except possibly one, have degree at most five.

Fellows, Kratochvíl, Middendorf, and Pfeiffer [8] give both polynomial-time solvable and NP-complete cases for the H -INDUCED MINOR CONTAINMENT problem. They also prove the following.

Theorem 1 ([8]). *For every fixed planar graph H , the H -INDUCED MINOR CONTAINMENT problem can be solved in polynomial time on planar input graphs.*

Brouwer and Veldman [3] initiated the research on the H -CONTRACTIBILITY problem. Their main result is stated below. A *dominating* vertex is a vertex adjacent to all other vertices.

Theorem 2 ([3]). *Let H be a connected triangle-free graph. The H -CONTRACTIBILITY problem can be solved in polynomial time if H has a dominating vertex, and is NP-complete otherwise.*

Note that a connected triangle-free graph with a dominating vertex is a star and that $H = P_4$ (path on four vertices) and $H = C_4$ (cycle on four vertices) are the smallest graphs H for which H -CONTRACTIBILITY is NP-complete. The research of Brouwer and Veldman [3] was continued by Levin et al. [14, 15].

Theorem 3 ([14, 15]). *Let H be a connected graph on at most five vertices. The H -CONTRACTIBILITY problem can be solved in polynomial time if H has a dominating vertex, and is NP-complete otherwise.*

The NP-completeness results in Theorems 2 and 3 can be extended using the notion of degree-two covers. Let $d_G(x)$ denote the degree of a vertex x in a graph G . A graph H' with an induced subgraph H is called a *degree-two cover* of H if the following two conditions both hold. First, for all $x \in V_H$, if $d_H(x) = 1$ then $d_{H'}(x) \geq 2$, and if $d_H(x) = 2$ and its two neighbors in H are adjacent then $d_{H'}(x) \geq 3$. Second, for all $x' \in V_{H'} \setminus V_H$, either x' has one neighbor and this neighbor is in H , or x' has two neighbors and these two neighbors form an edge in H .

Theorem 4 ([14]). *Let H' be a degree-two cover of a connected graph H . If H -CONTRACTIBILITY is NP-complete, then so is H' -CONTRACTIBILITY.*

In the papers by Brouwer and Veldman [3] and Levin et al. [14] several other results are shown. To discuss these we need some extra terminology (which we will use later in the paper as well). For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$, we denote their *join* by $G_1 \bowtie G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv \mid u \in V_1, v \in V_2\})$, and their *disjoint union* by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. For the disjoint union $G \cup G \cup \dots \cup G$ of k copies of the graph G , we write kG ; for $k = 0$ this yields the empty graph (\emptyset, \emptyset) . For integers $a_1, a_2, \dots, a_k \geq 0$, we let $H_i^*(a_1, a_2, \dots, a_k)$ be the graph $K_i \bowtie (a_1 P_1 \cup a_2 P_2 \cup \dots \cup a_k P_k)$, where K_i is the complete graph on i vertices and P_i is the path on i vertices. Note that $H_1^*(a_1)$ denotes a star on $a_1 + 1$ vertices. Brouwer and Veldman [3] show

that H -CONTRACTIBILITY can be solved in polynomial time for $H = H_1^*(a_1)$ or $H = H_1^*(a_1, a_2)$ for any $a_1, a_2 \geq 0$. Observe that $H_i^*(0) = K_i$ and that K_i -CONTRACTIBILITY is equivalent to K_i -MINOR CONTAINMENT, and hence solvable in polynomial time, by the previously mentioned result of Robertson and Seymour [17]. These results have been generalized by Levin et al. [14] leading to the following theorem.

Theorem 5 ([14]). *The H -CONTRACTIBILITY problem can be solved in polynomial time for:*

1. $H = H_1^*(a_1, a_2, \dots, a_k)$ for any $k \geq 1$ and $a_1, a_2, \dots, a_k \geq 0$
2. $H = H_2^*(a_1, a_2)$ for any $a_1, a_2 \geq 0$
3. $H = H_3^*(a_1)$ for any $a_1 \geq 0$
4. $H = H_i^*(0)$, for any $i \geq 1$.

Our Results and Paper Organization. In Section 2 we first recall some basic notions in parameterized complexity. Then we consider the INDUCED MINOR CONTAINMENT problem, where we assume that G belongs to some fixed *minor-closed* graph class \mathcal{G} (i.e., \mathcal{G} contains every minor of every member) and that H is planar. We prove that under these assumptions this problem becomes fixed parameter tractable in $|V_H|$. Since the class of planar graphs is minor-closed, this result generalizes Theorem 1.

The presence of a dominating vertex seems to play an interesting role in the complexity classification of the H -CONTRACTIBILITY problem. So far, in all polynomial-time solvable cases of this problem the pattern graph H has a dominating vertex, and in all NP-complete cases H does not have such a vertex. Following this trend, we extend Theorem 5 in Section 3.1 by showing that $H_4^*(a_1)$ -CONTRACTIBILITY can be solved in polynomial time for every $a_1 \geq 0$. In Section 3.2 however we present the first class of graphs H with a dominating vertex for which H -CONTRACTIBILITY is NP-complete. This result implies that the presence of a dominating vertex in the target graph H does not guarantee that the H -CONTRACTIBILITY problem can be solved in polynomial time (unless $\mathbf{P} = \mathbf{NP}$). However, it might still be the case that H -CONTRACTIBILITY is NP-complete whenever H does not have a dominating vertex. This motivates the study of the following variant of the H -CONTRACTIBILITY problems in Section 4.

(H, v) -CONTRACTIBILITY

Instance: A graph G and a positive integer k .

Question: Does G have an H -witness structure \mathcal{W} with $|W(v)| \geq k$?

The main result of Section 4 is a theorem stating that (H, v) -CONTRACTIBILITY is NP-complete whenever H is connected and v is not a dominating vertex of H . For example, let $P_3 = p_1 p_2 p_3$. Then the (P_3, p_3) -CONTRACTIBILITY problem is NP-complete (whereas P_3 -CONTRACTIBILITY can be solved in polynomial time). Section 5 contains the conclusions and mentions a number of open problems.

2 Induced Minors in Minor-Closed Classes

We start this section with a short introduction on the complexity classes XP and FPT . Both classes are defined in the framework of parameterized complexity as developed by Downey and Fellows [7]. The complexity class XP consists of parameterized decision problems Π such that for each instance (I, k) it can be decided in $\mathcal{O}(f(k)|I|^{g(k)})$ time whether $(I, k) \in \Pi$, where f and g are computable functions depending only on the *parameter* k , and $|I|$ denotes the size of I . So XP consists of parameterized decision problems which can be solved in polynomial time if the parameter is considered to be a constant. A problem is *fixed parameter tractable* in k if an instance (I, k) can be solved in time $\mathcal{O}(f(k)|I|^c)$, where f denotes a computable function and c a constant independent of k . Therefore, such an algorithm may provide a solution to the problem efficiently if the parameter is reasonably small. The complexity class $\text{FPT} \subseteq \text{XP}$ is the class of all fixed-parameter tractable decision problems.

We show that $\text{INDUCED MINOR CONTAINMENT}$ is fixed parameter tractable in $|V_H|$ on input pairs (G, H) with G from any fixed minor-closed graph class \mathcal{G} and H planar. Before doing this we first recall the following notions. A *tree decomposition* of a graph $G = (V, E)$ is a pair (\mathcal{X}, T) , where $\mathcal{X} = \{X_1, \dots, X_r\}$ is a collection of *bags*, which are subsets of V , and T is a tree on vertex set \mathcal{X} with the following three properties. First, $\bigcup_{i=1}^r X_i = V$. Second, for each $uv \in E$, there exists a bag X_i such that $\{u, v\} \subseteq X_i$. Third, if $v \in X_i$ and $v \in X_j$ then all bags in T on the (unique) path between X_i and X_j contain v . The *width* of a tree decomposition (\mathcal{X}, T) is $\max\{|X_i| - 1 \mid i = 1, \dots, r\}$, and the *treewidth* $\text{tw}(G)$ of G is the minimum width over all possible tree decompositions of G .

Our proof idea is as follows. We check if the input graph G has sufficiently large treewidth. If not, then we apply the monadic second-order logic result of Courcelle [6]. Otherwise, we show that G always contains H as an induced minor. Before going into details, we first introduce some additional terminology.

The $k \times k$ *grid* M_k has as vertex set all pairs (i, j) for $i, j = 0, 1, \dots, k - 1$, and two vertices (i, j) and (i', j') are joined by an edge if and only if $|i - i'| + |j - j'| = 1$. For $k \geq 2$, let Γ_k denote the graph obtained from M_k by triangulating its faces as follows: add an edge between vertices (i, j) and (i', j') if $i - i' = 1$ and $j' - j = 1$, and add an edge between corner vertex $(k - 1, k - 1)$ and every external vertex that is not already adjacent to $(k - 1, k - 1)$, i.e., every vertex (i, j) with $i \in \{0, k - 1\}$ or $j \in \{0, k - 1\}$, apart from the vertices $(k - 2, k - 1)$ and $(k - 1, k - 2)$. We let Π_k denote the graph obtained from Γ_k by adding a new vertex s that is adjacent to every vertex of Γ_k . See Figure 2 for the graphs M_6, Γ_6 , and Π_6 .

Let \mathcal{F} denote a set of graphs. Then a graph G is called \mathcal{F} -*minor-free* if G does not contain a graph in \mathcal{F} as a minor. If $\mathcal{F} = \{F\}$ we say that G is F -minor-free. We need the following results by Fomin et al. [9] and by Fellows et al. [8], respectively.

Theorem 6 ([9]). *For every graph F , there is a constant c_F such that every connected F -minor-free graph of treewidth at least $c_F \cdot k^2$ is Γ_k -contractible or Π_k -contractible.*

Theorem 7 ([8]). *For every planar graph H , there is a constant b_H such that every planar graph of treewidth at least b_H contains H as an induced minor.*

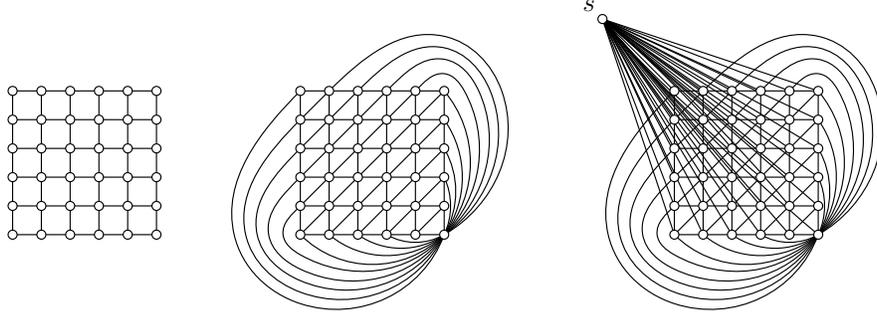


Figure 2: The graphs M_6 , Γ_6 , and Π_6 , respectively.

We also recall the well-known result of Robertson and Seymour [18] proving Wagner’s conjecture.

Theorem 8 ([18]). *A graph class \mathcal{G} is minor-closed if and only if there exists a finite set \mathcal{F} of graphs such that \mathcal{G} is equal to the class of \mathcal{F} -minor-free graphs.*

We are now ready to prove our generalization of Theorem 1. A graph class is *nontrivial* if it does not contain all graphs.

Theorem 9. *Let \mathcal{G} be any nontrivial minor-closed graph class. Then the INDUCED MINOR CONTAINMENT problem is fixed parameter tractable in $|V_H|$ on input pairs (G, H) with $G \in \mathcal{G}$ and H planar.*

Proof. Let H be a fixed planar graph with constant b_H as defined in Theorem 7. Let G be a graph on n vertices in a minor-closed graph class \mathcal{G} . From Theorem 8 we deduce that there exists a finite set \mathcal{F} of graphs such that G is \mathcal{F} -minor-free. Note that \mathcal{F} is nonempty, because \mathcal{G} is nontrivial. By Theorem 6, for each $F \in \mathcal{F}$, there exists a constant c_F such that every connected F -minor-free graph of treewidth at least $c_F \cdot b_H^2$ is Γ_{b_H} -contractible or Π_{b_H} -contractible. Let $c := \min\{c_F \mid F \in \mathcal{F}\}$. We first check if $\text{tw}(G) < c \cdot b_H^2$. We can do so as recognizing such graphs is fixed parameter tractable in $c \cdot b_H^2$ due to a result of Bodlaender [2].

Case 1. $\text{tw}(G) < c \cdot b_H^2$. The property of having H as an induced minor is expressible in monadic second-order logic (cf. [8]). Hence, by a well-known result of Courcelle [6], we can determine in $\mathcal{O}(|V_G|)$ time if G contains H as an induced minor.

Case 2. $\text{tw}(G) \geq c \cdot b_H^2$. We will show that in this case G is a yes-instance. By Theorem 6, we find that G is Γ_{b_H} -contractible or Π_{b_H} -contractible.

First suppose G is Γ_{b_H} -contractible. Then G has Γ_{b_H} as an induced minor. It is easy to prove that M_{b_H} has treewidth b_H . It is clear from the definition of treewidth that any supergraph of M_{b_H} , and Γ_{b_H} in particular, has treewidth at least b_H . Note that Γ_{b_H} is a planar graph. Then, by Theorem 7, Γ_{b_H} has H as an induced minor. Consequently, by transitivity, G has H as an induced minor.

Now suppose G is Π_{b_H} -contractible. Let \mathcal{W} be a Π_{b_H} -witness structure of G . We remove all vertices in $\mathcal{W}(s)$ from G . We then find that G has Γ_{b_H} as an induced minor and return to the previous situation. \square

3 The H -CONTRACTIBILITY Problem

As we mentioned in Section 1, the presence of a dominating vertex seems to play an interesting role in the complexity classification of the H -CONTRACTIBILITY problem. So far, in all polynomial-time solvable cases of this problem the pattern graph H has a dominating vertex, and in all NP-complete cases H does not have such a vertex. The first result of this section follows this pattern: we prove in Section 3.1 that $H_4^*(a_1)$ -CONTRACTIBILITY can be solved in polynomial time for every $a_1 \geq 0$. In Section 3.2 however we present the first class of graphs H with a dominating vertex for which H -CONTRACTIBILITY is NP-complete.

3.1 Polynomial Cases With Four Dominating Vertices

Let H and G be graphs such that G is H -contractible. Let \mathcal{W} be an H -witness structure of G . We call the subset of vertices in a witness set $W(h_i)$ that are adjacent to vertices in some other witness set $W(h_j)$ a *connector* $C_{\mathcal{W}}(h_i, h_j)$. We use the notion of connectors to simplify the witness structure of an $H_4^*(a_1)$ -contractible graph. Let y_1, \dots, y_4 denote the four dominating vertices of $H_4^*(a_1)$ and let x_1, \dots, x_{a_1} denote the remaining vertices of $H_4^*(a_1)$. For every $1 \leq i \leq a_1$, we define $C_{\mathcal{W}}(x_i, Y) := \bigcup_{j=1}^4 C_{\mathcal{W}}(x_i, y_j)$, and also call such a set a *connector*.

The graph $H_4^*(2)$ is shown in Figure 3, and two copies of an $H_4^*(2)$ -contractible graph G are shown in Figure 4. The dashed lines in the left and the right graph indicate two different $H_4^*(2)$ -witness structures \mathcal{W} and \mathcal{W}' of G , respectively. Exactly four vertices of the witness set $W(x_2)$ are adjacent to $W(y_1) \cup W(y_2) \cup W(y_3) \cup W(y_4)$, which means that those four vertices form the connector $C_{\mathcal{W}}(x_2, Y)$. When we consider the $H_4^*(2)$ -witness structure \mathcal{W}' of the right graph, we see that none of the connectors $C_{\mathcal{W}'}(x_1, Y)$ and $C_{\mathcal{W}'}(x_2, Y)$, formed by the grey vertices, contains more than two vertices.

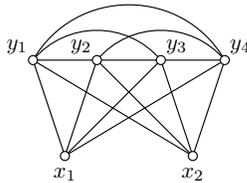


Figure 3: The graph $H_4^*(2)$.

The next lemma shows that every $H_4^*(a_1)$ -contractible graph has an $H_4^*(a_1)$ -witness structure \mathcal{W}' where every connector of the form $C_{\mathcal{W}'}(x_i, Y)$ has size at most two.

Lemma 1. *Let $a_1 \geq 0$. Every $H_4^*(a_1)$ -contractible graph has an $H_4^*(a_1)$ -witness structure \mathcal{W}' such that for every $1 \leq i \leq a_1$ one of the following two holds:*

- (i) $C_{\mathcal{W}'}(x_i, Y)$ consists of one vertex, and this vertex is adjacent to all four sets $W'(y_1)$, $W'(y_2)$, $W'(y_3)$, $W'(y_4)$;
- (ii) $C_{\mathcal{W}'}(x_i, Y)$ consists of two vertices, each of them adjacent to exactly two sets of $W'(y_1)$, $W'(y_2)$, $W'(y_3)$, $W'(y_4)$.

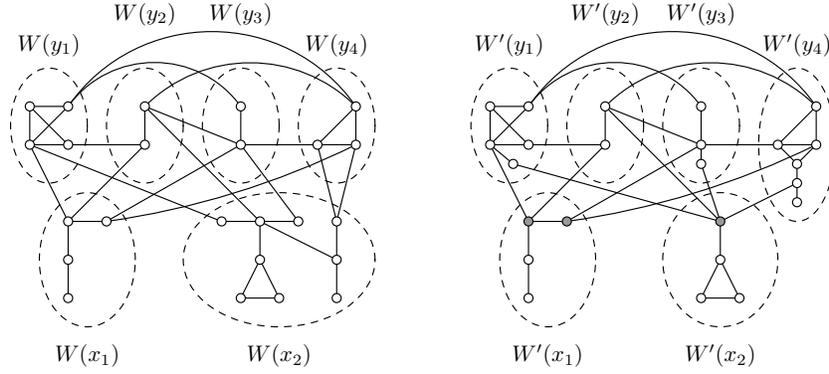


Figure 4: Two $H_4^*(2)$ -witness structures \mathcal{W} and \mathcal{W}' of a graph, where \mathcal{W}' is obtained from \mathcal{W} by moving as many vertices as possible from $W(x_1) \cup W(x_2)$ to $W(y_1) \cup W(y_2) \cup W(y_3) \cup W(y_4)$. The grey vertices form the connectors $C_{\mathcal{W}'}(x_1, Y)$ and $C_{\mathcal{W}'}(x_2, Y)$.

Proof. Let \mathcal{W} be an $H_4^*(a_1)$ -witness structure of an $H_4^*(a_1)$ -contractible graph G . Below we transform \mathcal{W} into a witness structure \mathcal{W}' that satisfies the statement of the lemma.

From each $W(x_i)$ we move as many vertices as possible to $W(y_1) \cup \dots \cup W(y_4)$ in a greedy way and without destroying the witness structure. This way we obtain an $H_4^*(a_1)$ -witness structure \mathcal{W}' of G . See Figure 4 for an example, where the $H_4^*(2)$ -witness structure \mathcal{W}' in the right graph is obtained from the $H_4^*(2)$ -witness structure \mathcal{W} on the left by performing this greedy procedure. We claim that $1 \leq |C_{\mathcal{W}'}(x_i, Y)| \leq 2$ for every $1 \leq i \leq a_1$.

Suppose, for contradiction, that $|C_{\mathcal{W}'}(x_i, Y)| \geq 3$ for some x_i . Let u_1, u_2, u_3 be three vertices in $C_{\mathcal{W}'}(x_i, Y)$. Let L_1, \dots, L_p denote the vertex sets of those components of $G[W'(x_i) \setminus \{u_1\}]$ that contain a vertex of $C_{\mathcal{W}'}(x_i, Y)$. Note that $p \geq 1$, because of the existence of u_2 and u_3 . Below we prove that $p = 1$ holds.

Observe that each L_q must be adjacent to at least two “unique” witness sets from $\{W'(y_1), \dots, W'(y_4)\}$, i.e., two witness sets that are not adjacent to $W'(x_i) \setminus L_q$, since otherwise we would have moved L_q to $W'(y_1) \cup \dots \cup W'(y_4)$. Since u_1 is adjacent to at least one witness set, this means that $p = 1$.

The fact that $p = 1$ implies that u_1 must be adjacent to at least two “unique” witness sets from $\{W'(y_1), \dots, W'(y_4)\}$, i.e., two witness sets that are not adjacent to $W'(x_i) \setminus \{u_1\}$; otherwise we would have moved u_1 and all components of $G[W'(x_i) \setminus \{u_1\}]$ not equal to L_1 to $W'(y_1) \cup \dots \cup W'(y_4)$. By the same arguments, exactly the same holds for u_2 and u_3 . This is not possible, as three vertices cannot be adjacent to two “unique” sets out of four. We conclude that $1 \leq |C_{\mathcal{W}'}(x_i, Y)| \leq 2$ for every $1 \leq i \leq a_1$.

Let $1 \leq i \leq a_1$. Suppose $|C_{\mathcal{W}'}(x_i, Y)| = 1$, say $C_{\mathcal{W}'}(x_i, Y) = \{p\}$. Then, by definition, p is adjacent to each of the four witness sets $W'(y_1), W'(y_2), W'(y_3), W'(y_4)$. Suppose $|C_{\mathcal{W}'}(x_i, Y)| = 2$, say $C_{\mathcal{W}'}(x_i, Y) = \{p, q\}$. Then p is adjacent to exactly two of the sets $W'(y_1), W'(y_2), W'(y_3), W'(y_4)$, and q is adjacent to the other two sets. In all other cases we would have moved p or q (and possibly some more vertices to keep all witness sets connected) to $W'(y_1) \cup \dots \cup W'(y_4)$. This completes the proof of Lemma 1. \square

We need one additional result, which can be found in the paper by Levin et al. [14], but follows directly from the polynomial-time result on minors by Robertson and Seymour [17].

Lemma 2 ([14]). *Let G be a graph and let $Z_1, \dots, Z_p \subseteq V_G$ be p specified non-empty pairwise disjoint sets such that $\sum_{i=1}^p |Z_i| \leq k$ for some fixed integer k . The problem of deciding whether G is K_p -contractible with K_p -witness sets U_1, \dots, U_p such that $Z_i \subseteq U_i$ for $i = 1, \dots, p$ can be solved in polynomial time.*

Recall that the problems $H_4^*(0)$ -CONTRACTIBILITY and $H_5^*(0)$ -CONTRACTIBILITY can be solved in polynomial time by Theorem 5. Since $H_5^*(0) = H_4^*(1)$, this means that $H_4^*(a_1)$ -CONTRACTIBILITY can be solved in polynomial time for $0 \leq a_1 \leq 1$. Using Lemma 1 and Lemma 2 we can generalize this as follows.

Theorem 10. *The $H_4^*(a_1)$ -CONTRACTIBILITY problem is solvable in polynomial time for any fixed non-negative integer a_1 .*

Proof. To test whether a connected graph G is $H_4^*(a_1)$ -contractible, we act as follows, due to Lemma 1. We guess a set $\mathcal{S} = \{C_{\mathcal{W}'}(x_i, Y) \mid 1 \leq i \leq a_1\}$ of connectors of size at most two. For each connector $C_{\mathcal{W}'}(x_i, Y)$ we act as follows.

If $C_{\mathcal{W}'}(x_i, Y)$ has size one, i.e., if $C_{\mathcal{W}'}(x_i, Y) = \{p\}$, then we guess four neighbors z_1, z_2, z_3, z_4 of p that are not contained in any connector of \mathcal{S} , and we put those vertices in sets Z_1, Z_2, Z_3, Z_4 , respectively. If a connector has size two, i.e., if $C_{\mathcal{W}'}(x_i, Y) = \{p, q\}$, then we guess two neighbors z_1, z_2 of p and two neighbors z_3, z_4 of q , such that all the vertices z_1, z_2, z_3, z_4 are different and none of them belongs to any of the connectors in \mathcal{S} ; we add vertex z_i to set Z_i for $i = 1, \dots, 4$. We then remove the vertices of every connector in \mathcal{S} from G and call the resulting graph G' .

We now check the following. First, we determine in polynomial time whether the set $Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ is contained in one component D of G' . If so, we check whether D is K_4 -contractible with K_4 -witness sets U_1, \dots, U_4 such that $Z_i \subseteq U_i$ for $i = 1, \dots, 4$. This can be done in polynomial time due to Lemma 2. If not, then we guess different sets of neighbors for the same set of connectors \mathcal{S} and repeat this step. Otherwise, we check whether the remaining components of G' together with the connectors $C_{\mathcal{W}'}(x_i, Y) \in \mathcal{S}$ form witness sets $W'(x_i)$ for $i = 1, \dots, a_1$. This can be done in polynomial time; there is only one unique way to do this, because witness sets $W'(x_i)$ are not adjacent to each other. If all possible sets of neighbors of the connectors in \mathcal{S} do not yield a positive answer, then we guess another set \mathcal{S} of connectors and start all over. As an example, see the right graph in Figure 4: if we guess the three grey vertices as set \mathcal{S} , and all of their neighbors in $W'(y_1) \cup \dots \cup W'(y_4)$ as the sets Z_1, \dots, Z_4 , then the algorithm described here would correctly decide that G is $H_4^*(2)$ -contractible.

Due to Lemma 1 the above algorithm is correct. Since we only have to guess $\mathcal{O}(n^{2a_1})$ sets \mathcal{S} with $\mathcal{O}(n^{4a_1})$ different sets of neighbors per set \mathcal{S} , and a_1 is fixed, it runs in polynomial time. \square

3.2 NP-Complete Cases With a Dominating Vertex

We show the existence of a class of graphs H with a dominating vertex such that H -CONTRACTIBILITY is NP-complete. To do this we need the following.

Proposition 11. *Let H be a graph. If H -INDUCED MINOR CONTAINMENT is NP-complete, then so are $(K_1 \bowtie H)$ -CONTRACTIBILITY and $(K_1 \bowtie H)$ -INDUCED MINOR CONTAINMENT.*

Proof. Let H and G be two graphs. We claim that the following three statements are equivalent.

- (i) G has H as an induced minor;
- (ii) $K_1 \bowtie G$ is $(K_1 \bowtie H)$ -contractible;
- (iii) $K_1 \bowtie G$ has $K_1 \bowtie H$ as an induced minor.

Below, we use G^* to denote the graph obtained from G by adding a new vertex x , and making x adjacent to every vertex of G . Similarly, H^* is the graph obtained from H by adding a new vertex y , and making y adjacent to every vertex of H . Note that G^* and H^* are isomorphic to the graphs $K_1 \bowtie G$ and $K_1 \bowtie H$, respectively.

“(i) \Rightarrow (ii)” Suppose G has H as an induced minor. Then, by definition, G contains an induced subgraph G' that is H -contractible. We extend an H -witness structure \mathcal{W} of G' to an H^* -witness structure of G^* by putting x and all vertices in $V_G \setminus V_{G'}$ in $W(y)$. This shows that G^* is H^* -contractible, or equivalently that $K_1 \bowtie G$ is $(K_1 \bowtie H)$ -contractible.

“(ii) \Rightarrow (iii)” Suppose $K_1 \bowtie G$ is $(K_1 \bowtie H)$ -contractible. By definition, $K_1 \bowtie G$ contains $K_1 \bowtie H$ as an induced minor.

“(iii) \Rightarrow (i)” Suppose G^* has H^* as an induced minor. Then G^* contains an induced subgraph G' that is H^* -contractible. Let \mathcal{W} be an H^* -witness structure of G' . Note that if $x \in V_{G'}$, then we may assume without loss of generality that $x \in W(y)$. We delete $W(y)$ and obtain an H -witness structure of the remaining subgraph of G' . This subgraph is an induced subgraph of G . Hence, G contains H as an induced minor. \square

Fellows et al. [8] showed that there exists a graph \bar{H} on 68 vertices such that \bar{H} -INDUCED MINOR CONTAINMENT is NP-complete; this graph is depicted in Figure 5. Combining their result with Proposition 11 (applied repeatedly) leads to the following corollary.

Corollary 12. *For any $i \geq 1$, $(K_i \bowtie \bar{H})$ -CONTRACTIBILITY is NP-complete.*

4 The (H, v) -CONTRACTIBILITY Problem

We start with an observation. A *star* is a complete bipartite graph in which one of the partition classes has size one. The unique vertex in this class is called the *center* of the star. We denote the star on $p + 1$ vertices with center c and leaves b_1, \dots, b_p by $K_{p,1}$.

Observation 1. *The $(K_{p,1}, c)$ -CONTRACTIBILITY problem can be solved in polynomial time for every $p \geq 1$.*

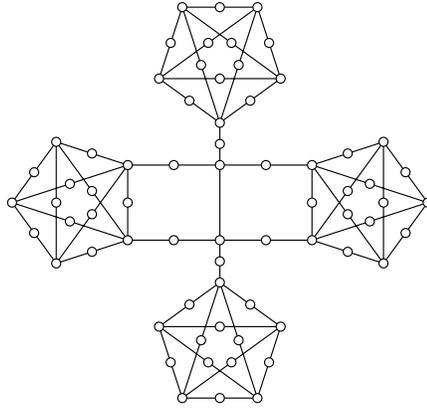


Figure 5: The graph \bar{H} .

Proof. Let graph $G = (V, E)$ and integer k form an instance of the $(K_{p,1}, c)$ -CONTRACTIBILITY problem. We may without loss of generality assume that $|V| \geq k + p$, since otherwise the answer is clearly negative. If G is $K_{p,1}$ -contractible, then there exists a $K_{p,1}$ -witness structure \mathcal{W} of G such that $|W(b_i)| = 1$ for all $1 \leq i \leq k$. This can be seen as follows. As long as $|W(b_i)| \geq 2$ we can move vertices from $W(b_i)$ to $W(c)$ without destroying the witness structure. Our algorithm would just guess the witness sets $W(b_i)$ and check whether $V \setminus (W(b_1) \cup \dots \cup W(b_p))$ induces a connected subgraph. As the total number of guesses is bounded by a polynomial in p , this algorithm runs in polynomial time. \square

The (H, v) -CONTRACTIBILITY problem takes as input a graph G and a parameter k . If $k = 1$, then the (H, v) -CONTRACTIBILITY problem is equivalent to the H -CONTRACTIBILITY problem, which leads to the following observation.

Observation 2. *Let H be a graph. If H -CONTRACTIBILITY is NP-complete, then (H, v) -CONTRACTIBILITY is NP-complete for every vertex $v \in V_H$.*

We expect that there are relatively few pairs (H, v) for which (H, v) -CONTRACTIBILITY can be solved in polynomial time (under the assumption $P \neq NP$). This is due to the Observation 2 and the following theorem, which is the main result of this section.

Theorem 13. *Let H be a connected graph and let v be a vertex of H . The (H, v) -CONTRACTIBILITY problem is NP-complete if v does not dominate H .*

Proof. Let H be a connected graph, and let v be a vertex of H that does not dominate H . Let $N_H(v)$ denote the neighborhood of v in H . We partition $V_H \setminus \{v\}$ into the following three sets

- $V_3 := V_H \setminus (N_H(v) \cup \{v\})$,
- $V_2 := \{w \in N_H(v) \mid w \text{ is not adjacent to } V_3\}$,
- $V_1 := \{w \in N_H(v) \mid w \text{ is adjacent to } V_3\}$.

Note that neither V_1 nor V_3 is empty because H is connected and v does not dominate H ; V_2 might be empty. In the top graph in Figure 7 a partition V_1, V_2, V_3 of the set $V_H \setminus \{v\}$ is depicted using dashed lines.

Clearly, (H, v) -CONTRACTIBILITY is in NP, because we can verify in polynomial time whether a given partition of the vertex set of a graph G forms an H -witness structure of G with $|W(v)| \geq k$. In order to show that (H, v) -CONTRACTIBILITY is NP-complete, we use a reduction from 3-SAT, which is well-known to be NP-complete (cf. [10]). Let $X = \{x_1, \dots, x_n\}$ be a set of variables and $C = \{c_1, \dots, c_m\}$ be a set of clauses making up an instance of 3-SAT. Let $\bar{X} := \{\bar{x} \mid x \in X\}$. We introduce two additional variables s and t , as well as $2n$ additional clauses $s_i := (x_i \vee \bar{x}_i \vee s)$ and $t_i := (x_i \vee \bar{x}_i \vee t)$ for $i = 1, \dots, n$. Let $S := \{s_1, \dots, s_n\}$ and $T := \{t_1, \dots, t_n\}$. Note that any truth assignment satisfies each of the $2n$ clauses in $S \cup T$. For every vertex $w \in V_1$ we create a copy X^w of the set X , and we write $X^w := \{x_1^w, \dots, x_n^w\}$. The literals s^w, t^w and the sets \bar{X}^w, C^w, S^w and T^w are defined similarly for every $w \in V_1$.

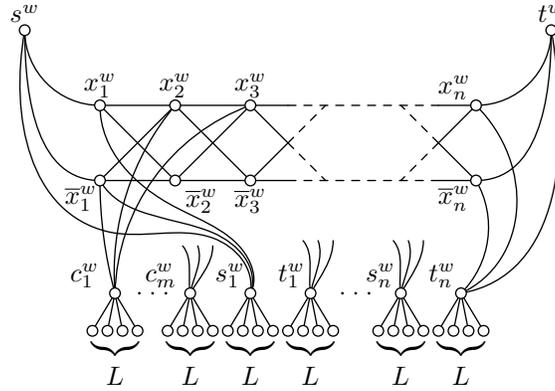


Figure 6: A subgraph G^w , where $c_1^w = (\bar{x}_1^w \vee x_2^w \vee x_3^w)$.

We construct a graph G such that C is satisfiable if and only if G has an H -witness structure \mathcal{W} with $|W(v)| \geq k$. In order to do this, we first construct a subgraph G^w of G for every $w \in V_1$ in the following way:

- every literal in $X^w \cup \bar{X}^w \cup \{s^w, t^w\}$ and every clause in $C^w \cup S^w \cup T^w$ is represented by a vertex in G^w
- we add an edge between $x \in X^w \cup \bar{X}^w \cup \{s^w, t^w\}$ and $c \in C^w \cup S^w \cup T^w$ if and only if x appears in c ;
- for every $i = 1, \dots, n - 1$, we add edges $x_i^w x_{i+1}^w$, $x_i^w \bar{x}_{i+1}^w$, $\bar{x}_i^w x_{i+1}^w$, and $\bar{x}_i^w \bar{x}_{i+1}^w$
- we add edges $s^w x_1^w$, $s^w \bar{x}_1^w$, $t^w x_n^w$, and $t^w \bar{x}_n^w$
- for every $c \in C^w \cup S^w \cup T^w$, we add L vertices whose only neighbor is c ; we determine the value of L later and refer to the L vertices as the *pendant vertices*.

See Figure 6 for a depiction of subgraph G^w . For clarity, most of the edges between the clause vertices and the literal vertices have not been drawn. We connect these $|V_1|$ subgraphs to each other as follows. For every $w, x \in V_1$, we add an edge between s^w and s^x in G if and only if w is adjacent to x in H . Let v^* be some fixed vertex in V_1 . We add an edge between $s_1^{v^*}$ and s_1^w for every $w \in V_1 \setminus \{v^*\}$. No other edges are added between vertices of two different subgraphs G^w and G^x . We add a copy of $H[V_2 \cup V_3]$ to G as follows. Vertex $x \in V_2$ is adjacent to s^w in G if and only if x is adjacent to w in H . Vertex $x \in V_3$ is adjacent to both s^w and t^w in G if and only if x is adjacent to w in H . Finally, we connect every vertex $x \in V_2$ to $s_1^{v^*}$. See Figure 7 for an example of a graph H and the graph G obtained from H by the procedure described above.

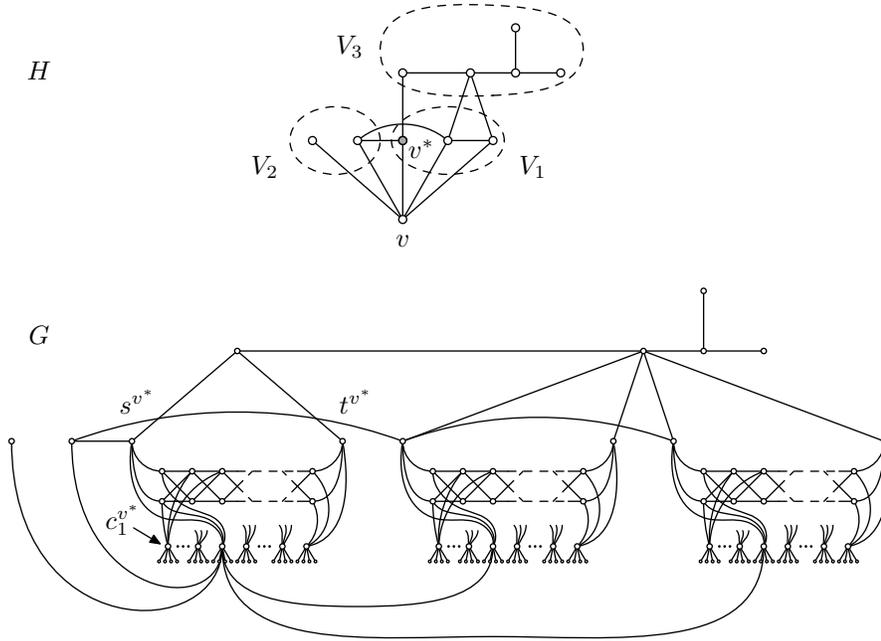


Figure 7: A graph H , where v^* is the grey vertex, and the corresponding graph G .

We define $L := (2 + 2n)|V_1| + |V_2| + |V_3|$ and $k := (L + 1)(m + 2n)|V_1|$. We prove that G has an H -witness structure \mathcal{W} with $|W(v)| \geq k$ if and only if C is satisfiable.

Suppose $\varphi : X \rightarrow \{T, F\}$ is a satisfying truth assignment for C . Let X_T (respectively X_F) be the variables that are set to true (respectively false) by φ . For every $w \in V_1$, we define $X_T^w := \{x_i^w \mid x_i \in X_T\}$ and $\overline{X}_T^w := \{\overline{x} \mid x \in X_T^w\}$; the sets X_F^w and \overline{X}_F^w are defined similarly. We define the H -witness sets of G as follows. Let $W(w) := \{w\}$ for every $w \in V_2 \cup V_3$, and let $W(w) := \{s^w, t^w\} \cup X_F^w \cup \overline{X}_T^w$ for every $w \in V_1$. Finally, let $W(v) := V_G \setminus (\bigcup_{w \in V_1 \cup V_2 \cup V_3} W(w))$. Note that for every $w \in V_1$ and for every $i = 1, \dots, n$, exactly one of x_i^w, \overline{x}_i^w belongs to $X_F^w \cup \overline{X}_T^w$. Hence, $G[W(w)]$ is connected for every $w \in V_1$. Since φ is a satisfying truth assignment for C , every c_i^w is adjacent to at least one vertex of $X_T^w \cup \overline{X}_F^w$ for every $w \in V_1$; by definition, this also holds for every s_i^w and t_i^w . This, together with the edges between $s_1^{v^*}$ and s_1^w for every $w \in V_1 \setminus \{v^*\}$, assures that $G[W(v)]$ is connected. So the witness set $G[W(w)]$ is connected for every $w \in V_H$. By construction, two witness sets $W(w)$ and $W(x)$ are adjacent if and only if w and x are adjacent

in H . Hence $\mathcal{W} := \{W(w) \mid w \in V_H\}$ is an H -witness structure of G . Witness set $W(v)$ contains $n|V_1|$ literal vertices, $(m + 2n)|V_1|$ clause vertices and L pendant vertices per clause vertex, i.e., $|W(v)| = (L + 1)(m + 2n)|V_1| + n|V_1| \geq k$.

In order to prove the reverse implication, suppose G has an H -witness structure \mathcal{W} with $|W(v)| \geq k$. We first show that all of the $(m + 2n)|V_1|$ clause vertices must belong to $W(v)$. Note that for every $w \in V_1$, the subgraph G^w contains $2 + 2n + (L + 1)(m + 2n)$ vertices: the vertices s^w and t^w , the $2n$ literal vertices in $X^w \cup \overline{X}^w$, the $m + 2n$ clause vertices and the $L(m + 2n)$ pendant vertices. Hence we have

$$|V_G| = (2 + 2n + (L + 1)(m + 2n))|V_1| + |V_2| + |V_3|.$$

Suppose there exists a clause vertex c that does not belong to $W(v)$. Then the L pendant vertices adjacent to c cannot belong to $W(v)$ either, as $W(v)$ is connected and the pendant vertices are only adjacent to c . This means that $W(v)$ can contain at most $|V_G| - (L + 1) = (L + 1)(m + 2n)|V_1| - 1$ vertices, contradicting the assumption that $W(v)$ contains at least $k = (L + 1)(m + 2n)|V_1|$ vertices. So all of the $(m + 2n)|V_1|$ clause vertices, as well as all the pendant vertices, must belong to $W(v)$.

We define $W_i := \bigcup_{w \in V_i} W(w)$ for $i = 1, 2, 3$ and prove four claims.

Claim 1: $V_3 = W_3$.

The only vertices of G that are not adjacent to any of the clause vertices or pendant vertices in $W(v)$ are the vertices of V_3 . As W_3 contains at least $|V_3|$ vertices, this proves Claim 1.

Claim 2: For any $w \in V_1$, both s^w and t^w belong to W_1 .

Let w be a vertex in V_1 , and let $w' \in V_3$ be a neighbor of w in H . Recall that both s^w and t^w are adjacent to w' in G . Suppose that s^w or t^w belongs to $W(v) \cup W_2$. By Claim 1, $w' \in W_3$. Then $W(v) \cup W_2$ and W_3 are adjacent. By construction, this is not possible. Suppose that s^w or t^w belongs to W_3 . Then W_3 and $W(v)$ are adjacent, as s^w and t^w are adjacent to at least one clause vertex, which belongs to $W(v)$. This is not possible.

Claim 3: For any $w \in V_1$, at least one of each pair x_i^w, \overline{x}_i^w of literal vertices belongs to $W(v)$.

Let $w \in V_1$. Suppose there exists a pair of literal vertices x_i^w, \overline{x}_i^w both of which do not belong to $W(v)$. Apart from its L pendant vertices, the vertex t_i^w is only adjacent to x_i^w, \overline{x}_i^w and t^w . The latter vertex belongs to W_1 due to Claim 2. Hence t_i^w and its L pendant vertices induce a component of $G[W(v)]$. Since $G[W(v)]$ contains other vertices as well, this contradicts the fact that $G[W(v)]$ is connected.

Claim 4: There exists a $w \in V_1$ for which at least one of each pair x_i^w, \overline{x}_i^w of literal vertices belongs to W_1 .

Let $S' := \{s^w \mid w \in V_1\}$ and $T' := \{t^w \mid w \in V_1\}$. By Claim 2, $S' \cup T' \subseteq W_1$. Suppose, for contradiction, that for every $w \in V_1$ there exists a pair x_i^w, \overline{x}_i^w of literal vertices, both of which do not belong to W_1 . Then for any $x \in V_1$, the witness set containing t^x does not contain any other vertex of $S' \cup T'$, as there is no path in $G[W_1]$ from t^x to any other vertex of $S' \cup T'$. But that means W_1 contains at least $|V_1| + 1$ witness sets, namely $|V_1|$ witness sets containing one vertex from T' , and at least one more witness set containing vertices of S' . This contradiction to the fact that W_1 , by definition, contains exactly $|V_1|$ witness sets finishes the proof of Claim 4.

Let $w \in V_1$ be a vertex for which of each pair x_i^w, \bar{x}_i^w of literal vertices exactly one vertex belongs to W_1 and the other vertex belongs to $W(v)$; such a vertex w exists as a result of Claim 3 and Claim 4. Let φ be the truth assignment that sets all the literals of $X^w \cup \bar{X}^w$ that belong to $W(v)$ to true and all other literals to false. Note that the vertices in C^w form an independent set in $W(v)$. Since $G[W(v)]$ is connected, each vertex $c_i^w \in C^w$ is adjacent to at least one of the literal vertices set to true by φ . Hence φ satisfies C . \square

5 Open Problems

The most challenging task is to finish the computational complexity classification of both the H -INDUCED MINOR CONTAINMENT problem and the H -CONTRACTIBILITY problem. With regards to the second problem, all previous evidence suggested some working conjecture stating that this problem can be solved in polynomial time if H contains a dominating vertex and NP-complete otherwise. However, in this paper we presented a class of graphs H with a dominating vertex for which H -CONTRACTIBILITY is NP-complete. This sheds new light on the H -CONTRACTIBILITY problem and raises a whole range of new questions.

1. *What is the smallest graph H that contains a dominating vertex for which H -CONTRACTIBILITY is NP-complete?*

The smallest graph known so far is the graph $K_1 \bowtie \bar{H}$, where \bar{H} is the graph on 68 vertices depicted in Figure 5. By Observation 2, we deduce that $(K_1 \bowtie \bar{H}, v)$ -CONTRACTIBILITY is NP-complete for all $v \in V_{K_1 \bowtie \bar{H}}$. This leads to the following question, which might be easier to answer than Question 1.

2. *What is the smallest graph H that contains a dominating vertex v for which (H, v) -CONTRACTIBILITY is NP-complete?*

We showed that (H, v) -CONTRACTIBILITY is NP-complete if H is connected and v does not dominate H . We still expect a similar result for H -CONTRACTIBILITY.

3. *Is the H -CONTRACTIBILITY problem NP-complete if H does not have a dominating vertex?*

Lemma 1 plays a crucial role in the proof of Theorem 10 that shows that $H_4^*(a_1)$ -CONTRACTIBILITY is polynomially solvable for every fixed a_1 . The lemma states that we can bound the size of connectors of the form $C_{\mathcal{W}^v}(x_i, Y)$ by a fixed constant, which guarantees that we only need to guess a polynomial number of sets in the proof of Theorem 10. Lemma 1 cannot be generalized such that it holds for the $H_i^*(a_1)$ -CONTRACTIBILITY problem for $i \geq 5$ and $a_1 \geq 2$. For example, there exist $H_5^*(2)$ -contractible graphs for which the size of the connectors $C_{\mathcal{W}^v}(x_i, Y)$ cannot be bounded by a constant. Hence, new techniques are required to attack the $H_i^*(a_1)$ -CONTRACTIBILITY problem for $i \geq 5$ and $a_1 \geq 2$. As a result of Theorem 5, the $H_5^*(a_1)$ -CONTRACTIBILITY problem can be solved in polynomial time for $0 \leq a_1 \leq 1$. It would be interesting to see whether we can find an analogue of Theorem 10 in case the target graph is $H_5^*(a_1)$.

4. *Is $H_5^*(a_1)$ -CONTRACTIBILITY solvable in polynomial time for every $a_1 \geq 0$?*

We expect that the (H, v) -CONTRACTIBILITY problem can be solved in polynomial time for only a few target pairs (H, v) . One such class of pairs might be (K_p, v) , where v is an arbitrary vertex of K_p . Using similar techniques as before (i.e., simplifying the witness structure), one can easily show that (K_p, v) -CONTRACTIBILITY can be solved in polynomial time for $p \leq 3$.

5. Is (K_p, v) -CONTRACTIBILITY solvable in polynomial time for every $p \geq 4$?

We finish this section with some remarks on fixing the parameter k in an instance (G, k) of the (H, v) -CONTRACTIBILITY problem.

Proposition 14. *The (P_3, p_3) -CONTRACTIBILITY problem is in XP.*

Proof. We first observe that any graph G that is a yes-instance of this problem has a P_3 -witness structure \mathcal{W} with $|W(p_1)| = 1$. This is so, as we can move all but one vertex from $W(p_1)$ to $W(p_2)$ without destroying the witness structure (see also Figure 1). Moreover, such a graph G contains a set $W^* \subseteq W(p_3)$ such that $|W^*| = k$ and $G[W^*]$ is connected. Hence we act as follows.

Let G be a graph. We guess a vertex v and a set V^* of size k . We put all neighbors of v in a set W_2 . We check if $G[V^*]$ is connected. If so, we check for each $y \in V_G \setminus (V^* \cup N(v) \cup \{v\})$ whether it is separated from $N(v)$ by V^* or not. If so, we put y in V^* . If not, we put y in W_2 . In the end we check if $G[W_2]$ and $G[V^*]$ are connected. If so, G is a yes-instance of (P_3, p_3) -CONTRACTIBILITY, as $W(p_1) = \{v\}$, $W(p_2) = W_2$ and $W(p_3) = V^*$ form a P_3 -witness structure of G with $|W(p_3)| \geq k$. If not, we guess another pair (v, V^*) and repeat the steps above. Since these steps can be performed in polynomial time and the total number of guesses is bounded by a polynomial in k , the result follows. \square

An affirmative answer to the next question would strengthen Proposition 14.

6. Is the (P_3, p_3) -CONTRACTIBILITY problem in FPT?

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