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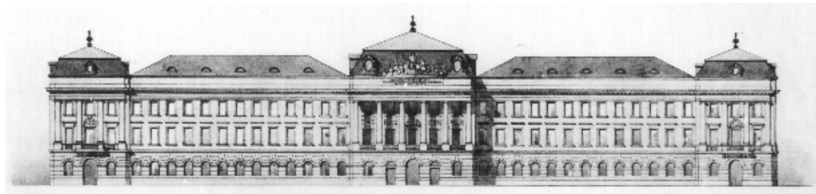

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ALGORITHMS AND COMPLEXITY RESULTS FOR PERSUASIVE ARGUMENTATION

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INFSYS RESEARCH REPORT 1843-11-03
JULY 2011

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ALGORITHMS AND COMPLEXITY RESULTS FOR PERSUASIVE ARGUMENTATION

Eun Jung Kim[1] Sebastian Ordyniak[2] Stefan Szeider[2]

Abstract. Value-based argumentation frameworks, as introduced by Bench-Capon, allow the abstract representation of persuasive argumentation. This formalism takes into account the relative strength of arguments with respect to some ordering which represents an audience. Deciding subjective or objective acceptance (i.e., acceptance with respect to at least one or with respect to all orderings) are intractable computational problems.

In this paper we study the computational complexity of testing the subjective or objective acceptance for problem instances that obey certain restrictions. We consider structural restrictions in terms of the underlying graph structure of the value-based argumentation framework and in terms of properties of the equivalence relation formed by arguments with the same relative strength. We identify new tractable fragments where subjective and objective acceptance can be tested in polynomial time. Furthermore we show the intractability of some fragments that are located at the boundary to tractability. Our results disprove two conjectures of Dunne (*Artificial Intelligence* 171, 2007).

Keywords: Value-based argumentation frameworks, treewidth, NP-hardness, polynomial-time tractability, subjective and objective acceptance.

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Acknowledgements: Ordyniak and Szeider's research was supported by the European Research Council, grant reference 239962 (COMPLEX REASON).

Publication Information: This is the author's self-archived copy of a paper that appeared in the proceedings of COMMA 2010, Third International Conference on Computational Models of Argument, Desenzano del Garda, Italy, September 8-10, 2010. Froniers in Artificial Intelligence and Applications, vol. 216, IOS Press, 2010, pp. 311-322. An extended version of this paper is to appear in *Artificial Intelligence*.

1 Introduction

The study of arguments as abstract entities and their interaction in form of *attacks* as introduced by Dung [8] has become one of the most active research branches within Logic, Artificial Intelligence, and Reasoning [4]. The key concept in this study is the notion of an abstract argumentation system or *argumentation framework* that can be considered as a directed graph whose nodes represent arguments, a directed edge that runs from an argument x to an argument y represents that "x attacks y."

Extending Dung's concept, Bench-Capon [2] introduced *value-based argumentation frameworks* (*VAFs*, for short) where arguments are ranked with respect to their strength, and an argument cannot attack another argument of higher rank. The ranking is specified by the combination of an assignment of values to arguments, and a (total) ordering of the values; the latter is called a (specific) *audience*. As explained by Bench-Capon [2], the role of arguments in this setting is to *persuade* rather than to prove, demonstrate or refute. An argument is said to be *subjectively accepted* in a value-based argumentation framework if it is accepted for at least one specific audience; it is *objectively accepted* if it is accepted for all specific audiences. Here acceptance refers to the standard semantics of *preferred extensions* [8].

Most computational problems that arise in the context of abstract argumentation are intractable [11]. In particular, as shown by Dunne and Bench-Capon [10], the problem SUBJECTIVE ACCEPTANCE (deciding whether an argument is subjectively accepted in a VAF) is NP-complete, and the problem OBJECTIVE ACCEPTANCE (deciding whether an argument is objectively accepted in a VAF) is co-NP-complete. In view of the intractability of these problems, it is a natural and relevant research question to ask for tractable fragments, and to carve out the boundaries between tractability and intractability.

In this paper we study the computational complexity of the problems SUBJECTIVE/OBJECTIVE ACCEPTANCE for VAFs that satisfy certain restrictions in terms of the following notions.

- Bounds on the largest number of arguments that share the same value; we call this parameter the *value-width* of the VAF under consideration.
- Bounds on the number of attacks (x, y) such that x and y share the same value; we call this parameter the *attack-width* of the VAF under consideration.
- Structural restrictions on the *graph structure* of the VAF under consideration; the graph structure is the graph whose vertices are arguments and where two distinct arguments are adjacent if and only if one attacks the other.
- Bounds on the *treewidth* of the *extended graph structure* of the VAF under consideration; the extended graph structure is obtained from the graph structure by adding edges between any two arguments that share the same value.

The parameters value-width and attack-width were first considered by Dunne [9] (using a different terminology). VAFs with a bipartite graph structure generalize the VAFs whose graph structure is a tree. To consider the treewidth of the extended graph structure is motivated by the fact that testing for subjective or objective acceptance is already intractable for VAFs whose graph structure is a tree [9].

Dunne [9] showed that SUBJECTIVE ACCEPTANCE remains NP-hard for instances of value-width 3 and whose graph structure is a tree. Furthermore, he stated the following two conjectures:

Conjecture 1 ([9]). SUBJECTIVE ACCEPTANCE *is polynomial-time decidable for VAFs of value-width* 2*.*

Conjecture 2 ([9]). SUBJECTIVE ACCEPTANCE *is fixed-parameter tractable when parameterized by the attack-width.*

A decision problem is called *fixed-parameter tractable* if an instance of size n and parameter k can be decided in time $O(f(k)n^c)$ where f is an arbitrary computable function and c is a constant independent of k [7]. Note that if a problem is fixed-parameter tractable, then each "slice" of the problem (obtained by fixing k to a constant) is decidable in polynomial time [12].

New Results

We have obtained positive and negative complexity results for SUBJECTIVE/OBJECTIVE ACCEPTANCE.

Theorem 1. (A) SUBJECTIVE ACCEPTANCE remains NP-hard for instances of value-width 2 and attack-width 1.

(B) OBJECTIVE ACCEPTANCE remains co-NP-hard for instances of value-width 2 and attack-width 1.

As a consequence of (A), Conjectures 1 and 2 do not hold unless P $=$ NP. It is easy to see that SUBJECTIVE/OBJECTIVE ACCEPTANCE become trivial if we further restrict the value-width to 1 or if we further restrict the attack-width to 0 [9], hence the bounds in Theorem 1 are tight.

On the positive side we show that Conjecture 1 is true if we restrict ourselves to VAFs with a bipartite graph structure.

Theorem 2. (A) SUBJECTIVE ACCEPTANCE can be decided in polynomial time for instances with a bipartite graph structure and of value-width 2.

(B) OBJECTIVE ACCEPTANCE can be decided in polynomial time for instances with a bipartite graph structure and of value-width 2.

Since trees are bipartite, it follows that both problems can be decided in polynomial time for VAFs of value-width 2 if the graph structure is a tree.

Finally, we consider VAFs with value-width ≥ 2 and an *extended graph structure* of bounded treewidth.

Theorem 3. SUBJECTIVE/OBJECTIVE ACCEPTANCE *can be decided in linear time for instances whose extended graph structure has bounded treewidth. This remains true even for instances whose value-width is greater than two.*

We obtain this result by expressing the problems within the formalism of monadic second-order (MSO) logic on finite structures, and using Courcelle's meta-theorem [6, 12].

2 Preliminaries

In this section we introduce the objects of our study more formally.

An *abstract argumentation system* or *argumentation framework* (*AF*, for short) is a pair (X, A) where X is a finite set of elements called *arguments* and $A \subseteq X \times X$ is a binary relation called the *attack relation*. If $(x, y) \in A$ we say that x *attacks* y.

An AF $F = (X, A)$ can be considered as a directed graph, and therefore it is convenient to borrow notions and notation from graph theory. For example, if $(x, y) \in A$ then we say that x is an in-neighbor of y and that y is an out-neighbor of x. We write $N_F^-(x)$ and $N_F^+(x)$ for the sets of in- respectively out-neighbors of x in F, and we omit the subscript if F is clear from the context.

Next we define commonly used semantics of AFs as introduced by Dung [8] (for the discussion of other semantics and variants, see e.g., Baroni and Giacomin's survey [1]). Let $F = (X, A)$ be an AF and $S \subseteq X$.

1. S is *conflict-free* in F if there is no $(x, y) \in A$ with $x, y \in S$.
2. S is *acceptable* in F if for each $x \in S$ and $y \in N^-(x)$ we have $N^-(y) \cap S \neq \emptyset$.
3. S is *admissible* in F if it is conflict-free and acceptable.
4. S is a *preferred extension* of F if S is admissible in F and there is no admissible set S' of F that properly contains S.

Let $F = (X, A)$ be an AF and $x_1 \in X$. The argument x_1 is *credulously accepted* in F if x_1 is contained in some preferred extension of F, and x_1 is *skeptically accepted* in F if x_1 is contained in all preferred extensions of F. In this paper we are especially interested in finding preferred extensions in acyclic AFs. It is well known that every acyclic AF $F = (X, A)$ has a unique preferred extension S_F, and that S_F can be found in polynomial time (S_F coincides with the "grounded extension" [8]). In fact, S_F can be found via a simple labeling procedure that repeatedly applies the following two rules to the arguments in X until each of them is either labeled IN or OUT: (1) An argument x is labeled IN if all in-neighbors of x are labeled OUT. (2) An argument x is labeled OUT if there exists an in-neighbor of x with label IN. The unique preferred extension S_F is then the set of all arguments that are labeled IN.

A *value-based argumentation framework* (VAF) is a tuple $F = (X, A, V, \eta)$ where (X, A) is an argumentation framework, V is a set of *values* and η is a mapping $X \to V$ such that the graph $(\eta^{-1}(v), \{\, (x, y) \in A \mid x, y \in \eta^{-1}(v) \,\})$ is acyclic for all $v \in V$. An *audience* $\leq$ for a VAF is a partial ordering $\leq$ on the set of values of F. Given a VAF $F = (X, A, V, \eta)$ and an audience $\leq$ for F, we define the AF $F_{\leq} = (X, A_{\leq})$ by setting $A_{\leq} = \{\, (x, y) \in A \mid \neg(\eta(x) < \eta(y)) \,\}$. An audience $\leq$ is *specific* if it is a total ordering on V. For an audience $\leq$ we also define $<$ in the obvious way, i.e., $x < y$ if and only if $x \leq y$ and $x \neq y$. Note that if $\leq$ is a specific audience, then $F_{\leq} = (X, A_{\leq})$ is an acyclic digraph and thus, has a unique preferred extension [4]. For a VAF $F = (X, A, V, \eta)$ and a value $v \in V$ we denote by $F - v$ the VAF obtained from F by deleting all arguments with value v and all attacks involving these arguments.

Let $F = (X, A, V, \eta)$ be a VAF. We say that an argument $x_1 \in X$ is *subjectively accepted in* F if there exists a specific audience $\leq$ such that x_1 is in the unique preferred extension of $F_{\leq}$. Similarly, we say that an argument $x_1 \in X$ is *objectively accepted in* F if x_1 is contained in the unique preferred extension of $F_{\leq}$ for every specific audience $\leq$.

We consider the following decision problems.

SUBJECTIVE ACCEPTANCE

Instance: A VAF $F = (X, A, V, \eta)$ and an argument $x_1 \in X$.

Question: Is x_1 subjectively accepted in F?

OBJECTIVE ACCEPTANCE

Instance: A VAF $F = (X, A, V, \eta)$ and an argument $x_1 \in X$.

Question: Is x_1 objectively accepted in F?

Considering an instance (F, x_1) of SUBJECTIVE/OBJECTIVE ACCEPTANCE, we shall refer to the argument x_1 as the *initial argument*.

Let $F = (X, A, V, \eta)$ be a VAF. We define the *value-width* of F as the largest number of arguments with the same value, i.e., $\max_{v \in V} |\eta^{-1}(v)|$, and the *attack-width* as the cardinality of the set $\{ (x, y) \in A \mid \eta(x) = \eta(y) \}$. The *graph structure* of F is the undirected graph $G_F = (X, E)$ where $E := \{ \{u, v\} \mid (u, v) \in A \}$. We say that a VAF F is a *tree* if G_F is a tree. Similarly we say that F is *bipartite* if G_F is a bipartite graph.

3 Certifying Paths

In this section we introduce the notion of a certifying path that is key for the proofs of Theorems 1 and 2.

Let $F = (X, A, V, \eta)$ be a VAF of value-width 2. We call an odd-length sequence $C = (x_1, z_1, \ldots, x_k, z_k, t)$, $k \geq 0$, of distinct arguments a *certifying path for* $x_1 \in X$ *in* F if it satisfies the following conditions:

C1 For every $1 \leq i \leq k$ it holds that $\eta(z_i) = \eta(x_i)$.

C2 For every $1 \leq i \leq k$ there exists a $1 \leq j \leq i$ such that $(z_i, x_j) \in A$.

C3 For every $2 \leq i \leq k$ it holds that $(x_i, z_{i-1}) \in A$ and $N_F^+(x_i) \cap \{z_i, x_1, \ldots, x_{i-1}\} = \emptyset$.

C4 $(t, z_k) \in A$ and $N_F^+(t) \cap \{x_1, \ldots, x_k\} = \emptyset$.

C5 If there exists a $z \neq t$ with $\eta(z) = \eta(t)$ then either $(t, z) \in A$ or $N_F^+(z) \cap \{x_1, \ldots, x_k, t\} = \emptyset$.

Lemma 1. *Let $F = (X, A, V, \eta)$ be a VAF of value-width 2 and $x_1 \in X$. Then x_1 is subjectively accepted in F if and only if there exists a certifying path for x_1 in F.*

Proof. Let $C = (x_1, z_1, \dots, x_k, z_k, t)$ be a certifying path for x_1 in F. Take a specific audience $\leq$ such that $\eta(x_1) < \cdots < \eta(x_k) < \eta(t)$ and all other values in V are smaller than $\eta(x_1)$. We claim that the unique preferred extension P of $F_{\leq}$ includes $\{x_1, \dots, x_k, t\}$ and excludes $\{z_1, \dots, z_k\}$, which means that x_1 is subjectively accepted in F. It follows from C5 that t is not attacked by any other argument in $F_{\leq}$ and hence $t \in P$ (see also Section 2 for a description of an algorithm to find the unique preferred extension of an acyclic AF). From C4 it follows that $z_k \notin P$. Furthermore, if there exists an argument $z \neq t$, $\eta(t) = \eta(z)$ then either $(t, z) \in A_{\leq}$ or there is no arc from z to an argument in $\{x_1, \dots, x_k, t\}$. In the first case $z \notin P$ and does not influence the membershship in P for any other arguments in X. In the second case $z \in P$ but there are no arcs to any argument in $\{x_1, \dots, x_k, t\}$. In both cases it follows that $x_k \in P$. Using C3 it follows that $z_{k-1} \notin P$ and since we already know that $z_k \notin P$ it follows that $x_{k-1} \in P$. A repeated application of the above arguments establishes the claim, and hence $x_1 \in P$ follows.

Conversely, suppose that there exists a specific audience $\leq$ such that x_1 is contained in the unique preferred extension P of $F_{\leq}$. We will now construct a certifying path C for x_1 in F. Clearly, if there is no $z_1 \in X \setminus \{x_1\}$ with $\eta(z_1) = \eta(x_1)$ and $(z_1, x_1) \in A$, then (x_1) is a certifying path for x_1 in F. Hence, it remains to consider the case where such a z_1 exists. Since $x_1 \in P$ it follows that $z_1 \notin P$. The sequence (x_1, z_1) clearly satisfies properties C1–C3. We now show that we can always extend such a sequence until we have found a certifying path for x_1 in F. Hence, let $S = (x_1, z_1, \dots, x_l, z_l)$ be such a sequence satisfying conditions C1–C3, and in addition assume S satisfies the following two conditions:

S1 It holds that $\eta(x_1) < \cdots < \eta(x_l)$.

S2 For every $1 \leq i \leq l$ we have $x_i \in P$ and $z_i \notin P$.

Clearly, the sequence (x_1, z_1) satisfies S1 and S2, hence we can include these conditions in our induction hypothesis. It remains to show how to extend S to a certifying path. Let $Z := \{\, x' \in P \mid (x', z_l) \in A \wedge \eta(x') > \eta(x_l) = \eta(z_l) \,\}$. Then $Z \neq \emptyset$ because $z_\ell \notin P$ by condition S2 and the assumption that P is a preferred extension. If there is a $t \in Z$ such that $(x_1, z_1, \dots, x_l, z_l, t)$ is a certifying path for x_1 in F we are done. Hence assume there is no such $t \in Z$.

We choose $x_{l+1} \in Z$ arbitrarily. Note that C' satisfies the condition C4; $(x_{l+1}, z_l) \in A$ (as $x_{l+1} \in Z$) and $(x_{l+1}, x_i) \notin A$ for $1 \leq i \leq l$ (as $x_{l+1}, x_i \in P$ and P is conflict-free). Since we assume that C' is not a certifying path, C' must violate C5.

It follows that there exists some $z_{l+1} \in X$ with $\eta(z_{l+1}) = \eta(x_{l+1})$ such that $(x_{l+1}, z_{l+1}) \notin A$ and $(z_{l+1}, x_i) \in A$ for some $1 \leq i \leq l+1$. We conclude that $S' = (x_1, z_1, \dots, x_l, z_l, x_{l+1}, z_{l+1})$ satisfies conditions C1–C3 and S1–S2. Hence, we are indeed able to extend S and will eventually obtain a certifying path for x_1 in F. □

Lemma 2. *Let $F = (X, A, V, \eta)$ be a VAF of value-width 2 and $x_1 \in X$. Then x_1 is objectively accepted in F if and only if for every $p \in N_F^-(x_1)$ it holds that $\eta(p) \neq \eta(x_1)$ and p is not subjectively accepted in $F - \eta(x_1)$.*

Proof. Assume that x_1 is objectively accepted in F. Suppose there is a $p \in N_F^-(x)$ with $\eta(p) = \eta(x_1)$. If we take a specific audience $\leq$ where $\eta(x_1)$ is the greatest element, then x_1 is not in the unique preferred extension of $F_{\leq}$, a contradiction to the assumption that x_1 is objectively accepted. Hence $\eta(p) \neq \eta(x_1)$ for all $p \in N_F^-(x_1)$. Next suppose there is a $p \in N_F^-(x_1)$ that is subjectively accepted in $F - \eta(x_1)$. Let $\leq$ be a specific audience such that p is in the unique preferred extension of $(F - \eta(x_1))_{\leq}$. We extend $\leq$ to a total ordering of V ensuring $\eta(x_1) \leq \eta(p)$. Clearly x_1 is not in the unique preferred extension of $F_{\leq}$, again a contradiction. Hence indeed for all $p \in N_F^-(x_1)$ we have $\eta(p) \neq \eta(x_1)$ and p is not subjectively accepted in $F - \eta(x_1)$

We establish the reverse direction by proving its counter positive. Assume that x_1 is not objectively accepted in F. We show that there exists some $p \in N_F^-(x_1)$ such that either $\eta(p) = \eta(x_1)$ or p is subjectively accepted in $F - \eta(x_1)$. Let $\leq$ be a specific audience of F such that x_1 is not in the unique preferred extension P of $F_{\leq}$. In view of the labeling procedure for finding P as sketched in Section 2, it follows that there exists some $p \in N_F^-(x_1) \cap P$ with $\eta(x_1) \leq \eta(p)$. If $\eta(x_1) = \eta(p)$ then we are done. On the other hand, if $\eta(p) \neq \eta(x_1)$, then p is in the unique preferred extension of $(F - \eta(x_1))_{\leq}$, and so p is subjectively accepted in $F - \eta(x_1)$. □

4 Proof of Hardness Results

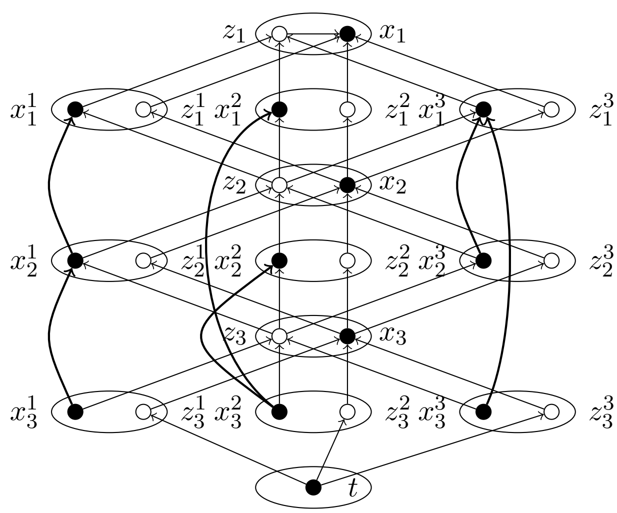

Figure 1: The instance F in the proof of Theorem 1 for the 3-CNF Formula $(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$.

Proof of Theorem 1(A). We devise a polynomial reduction from 3-SAT. Let Φ be a 3-CNF formula with clauses $C_1, \ldots, C_m$ and $C_j = x_{j,1} \vee x_{j,2} \vee x_{j,3}$ for every $1 \leq j \leq m$. We construct a VAF $F = (X, A, V, \eta)$ of value-width 2 and attack-width 1 such that the initial argument $x_1 \in X$ is subjectively accepted in F if and only if Φ is satisfiable. See Figure 1 for an example.

The set X contains: (i) two vertices x_j, z_j for every clause C_j; (ii) two vertices x_j^i, z_j^i for every clause C_j and $1 \leq i \leq 3$; (iii) one vertex t.

The set A contains: (i) one arc from z_1 to x_1; (ii) one arc from x_j^i to z_j and one arc from z_j^i to x_j for every $1 \leq j \leq m$ and $1 \leq i \leq 3$; (iii) one arc from x_{j+1} to z_j^i and one arc from z_{j+1} to x_j^i for every $1 \leq j < m$ and $1 \leq i \leq 3$; (iv) one arc from t to z_m^i for every $1 \leq i \leq 3$; (v) one arc from x_j^i to $x_{j'}^{i'}$ for every $1 \leq j' < j \leq m$ and $1 \leq i, i' \leq 3$ such that $x_{j,i}$ and $x_{j',i'}$ are complementary literals.

Let $V = \{ v_j \mid 1 \leq j \leq m \} \cup \{ v_j^i \mid 1 \leq j \leq m \wedge 1 \leq i \leq 3 \} \cup \{v_t\}$ be the set of values, $|V| = 4m + 1$, and let x_1 be the initial argument. The mapping η is defined such that $\eta(x_j) = \eta(z_j) = v_j$, $\eta(x_j^i) = \eta(z_j^i) = v_j^i$ for every $1 \leq j \leq m$ and $1 \leq i \leq 3$, and $\eta(t) = v_t$. It is easy to see that F has attack-width 1 and value-width 2 and can be constructed from Φ in polynomial time. It remains to show that Φ is satisfiable if and only if x_1 is subjectively accepted in F.

To see this note that every certifying path for x_1 in F must have the form $(x_1, z_1, x_1^{i_1}, z_1^{i_1}, x_2, z_2, x_2^{i_2}, z_2^{i_2}, x_3, z_3, \dots, x_m, z_m, x_m^{i_m}, z_m^{i_m}, t)$ such that $i_j \in \{1, 2, 3\}$ for every $1 \leq j \leq m$ and for every pair $1 \leq j < j' \leq m$ there is no arc from a vertex $x_{j'}^{i_{j'}}$ to a vertex $x_j^{i_j}$. Hence there exists a certifying path for x_1 in F if and only if there exists a set L of literals that corresponds to a satisfying truth assignment of Φ (i.e., L contains a literal of each clause of Φ but does not contain a complementary pair of literals). □

Proof of Theorem 1(B). Let F be the VAF as constructed in the proof of part (A) of this theorem. Let $F' = (X', A', V', \eta')$ be the VAF such that $X' := X \cup \{x_0\}$, $A' := A \cup \{(x_1, x_0)\}$, $V' := V \cup \{v_0\}$, $\eta'(x_0) = v_0$ and $\eta'(x) = \eta(x)$ for every $x \in X$. Then it is easy to see that x_0 is objectively accepted in F' if and only if Φ is not satisfiable. □

5 Polynomial-Time Algorithm for Bipartite VAFs

This subsection is devoted to prove Theorem 2. Throughout this section, we assume that we are given a bipartite VAF $F = (X, A, V, \eta)$ together with an initial argument x_1. Furthermore, let X_{even} and X_{odd} be the subsets of X containing all arguments x such that the length of a shortest directed path in F from x to x_1 is even respectively odd.

Lemma 3. *Let $C = (x_1, z_1, \dots, x_k, z_k, t)$ be a certifying path for x_1 in F. Then $(\{ x_i \mid 1 \leq i \leq k \} \cup \{t\}) \subseteq X_{\text{even}}$ and $\{ z_i \mid 1 \leq i \leq k \} \subseteq X_{\text{odd}}$.*

Proof. The claim follows easily via induction on k by using the properties of a certifying path and the fact that F is bipartite. □

Based on the observation of Lemma 3, we construct an auxiliary directed graph $H_F := (V, E)$ as follows. The vertex set of H_F is the set of values V of F and there is an arc from $v_i \in V$ to $v_j \in V$ if there is an argument $x \in X_{\text{even}}$ with $\eta(x) = v_i$ and an argument z with $\eta(z) = v_j$ such that $(x, z) \in A$. Note that $z \in X_{\text{odd}}$ since F is bipartite.

Lemma 4. *If $C = (x_1, z_1, \dots, x_k, z_k, t)$ is a certifying path for x_1 in F, then $(\eta(t), \eta(x_k), \dots, \eta(x_1))$ is a directed path from $\eta(t)$ to $\eta(x_1)$ in H_F.*

Proof. By the definition of a certifying path, we have $(t, z_k) \in A$ and for every $2 \leq i \leq k$ it holds that $(x_i, z_{i-1}) \in A$. Lemma 3 implies that t and x_i for every $1 \leq i \leq k$ are contained in X_{even} and hence $(\eta(t), \eta(x_k)) \in E$ and $(\eta(x_i), \eta(x_{i-1})) \in E$ for every $1 < i \leq k$. □

Lemma 4 tells us that we can limit ourselves in searching for a directed path in H_F in order to find a certifying path for x_1 in F. We want to know exactly what kind of directed path will correspond to a certifying path. To this end, we consider a subgraph $H_F^{-v_i}$ of H_F for each $v_i \in V$ obtained as follows: if there is an argument $z_i \in X_{\text{odd}}$ with $\eta(z_i) = v_i$ and there is no argument $x_i \in \eta^{-1}(v_i) \setminus \{z_i\}$ with $(x_i, z_i) \in A$, then remove every vertex $\eta(y_i)$ from H_F such that $y_i \in N_F^+(z_i)$ and $y_i \in X_{\text{even}}$. Note that if there is a $y_i \in N_F^+(z_i)$ with $\eta(y_i) = \eta(z_i)$, then we remove the vertex v_i itself from H_F.

Lemma 5. *Consider an odd-length sequence $C = (x_1, z_1, \ldots, x_k, z_k, t)$ of distinct arguments of a bipartite VAF F of value width 2. Then C is a certifying path for x_1 in F if and only if the following conditions hold:*

(1) $\eta(x_i) = \eta(z_i)$ *for* $1 \leq i \leq k$.

(2) $(\eta(t), \eta(x_k), \ldots, \eta(x_1))$ *is a directed path from* $\eta(t)$ *to* $\eta(x_1)$ *in* $H_F^{-\eta(t)}$.

(3) None of the sub-sequences $\eta(x_i), \ldots, \eta(x_1)$ *is a directed path from* $\eta(x_i)$ *to* $\eta(x_1)$ *in* $H_F^{-\eta(x_i)}$ *for* $1 \leq i \leq k$.

Proof. Assume $C = (x_1, z_1, \ldots, x_k, z_k, t)$ is a certifying path for x_1 in F. Property (1) follows from condition C1 of a certifying path, property (2) follows from condition C5 and Lemma 4. Property (3) follows from conditions C2 and C3.

To see the reverse assume that C satisfies properties (1)–(3). Condition C1 follows from property (1). Conditions C3, C4 and C5 follow from property (2) and the assumption that F is bipartite. Condition C2 follows from property (3). Hence C is a certifying path for x_1 in F. □

Lemma 5 suggests a simple strategy to find a certifying path for x_1 in F, if one exists. If for some $v_t \in V$ there exists a directed path P from v_t to $\eta(x_1)$ in $H_F^{-v_t}$ and v_t is the closest value to $\eta(x_1)$ with this property, then the sequence of arguments in X whose values form P is a certifying path for x_1 in F. On the other hand, if there is no such value, then there is no certifying path for x_1 in F.

We call this algorithm DETECT CERTIFYING PATH, summarized below.

1. For each $v \in V$, we check whether there is a directed path from v to $v_1 = \eta(x_1)$ in H_F^{-v}. Find a shortest path, if one exists.

2. If there exists a vertex v which has a shortest directed path $v, v_k, \ldots, v_1$ in H_F^{-v}, then among such vertices choose one with minimum k. Take the total ordering $<$ as $v_1 < \cdots < v_k < v$ and $v' < v_1$ for every $v' \in V \setminus \{v, v_1, \ldots, v_k\}$.

3. If there is no such vertex, return NO.

Proposition 1. *The algorithm* DETECT CERTIFYING PATH *correctly returns a certifying path for* x_1 *if one exists and returns NO otherwise in time* $O(|V| \cdot (|V| + |E|))$.

Proof. The correctness of DETECT CERTIFYING PATH follows from Lemma 5. For each $v \in V$, building H_F^{-v} and finding a shortest directed path from v to $v_1 = \eta(x_1)$, if one exists, takes $O(|V| + |E|)$ time. As we iterate over all vertices of V, the claimed running time follows. □

Proof of Theorem 2. Statement (A) of the theorem follows from Lemma 1 and Proposition 1. Statement (B) follows from Statement (A) and Lemma 2. □

6 Linear-Time Algorithm for VAFs of Bounded Treewidth

As mentioned above, it is known that SUBJECTIVE/OBJECTIVE ACCEPTANCE are intractable even when the given VAF is a tree. This is perhaps not surprising since two arguments of a tree-like VAF can be considered as linked with each other because they have the same value. In fact, such links may form cycles in an otherwise tree-like VAF. Therefore we propose to consider the *extended graph structure* of a VAF that takes such links into account. We show that SUBJECTIVE/OBJECTIVE ACCEPTANCE are easy for VAFs whose extended graph structure is a tree, and more generally, the problem can be solved in linear-time for VAFs with an extended graph structure of bounded treewidth (treewidth is a popular graph parameter that indicates in a certain sense how similar a graph is to a tree; we give a definition of treewidth below).

Let $F = (X, A, V, \eta)$ be a VAF. We define the *extended graph structure* of F as the graph $G = (X, E)$ where E contains an edge between two distinct arguments $x, y \in X$ if and only if $(x, y) \in A$ or $(y, x) \in A$ or $\eta(x) = \eta(y)$. We define the treewidth of a VAF as the treewidth of its extended graph structure.

The treewidth of a graph is defined using the following notion of a tree decomposition [5]: a *tree decomposition* of $G = (V, E)$ is a pair (T, χ) where T is a tree and χ is a labeling function with $\chi(t) \subseteq V$ for every tree node t such that the following conditions hold: (i) Every vertex of G occurs in $\chi(t)$ for some tree node t. (ii) For every edge $\{u, v\}$ of G there is a tree node t such that $u, v \in \chi(t)$. (iii) For every vertex v of G, the tree nodes t with $v \in \chi(t)$ induce a connected subtree of T. The *width* of a tree decomposition (T, χ) is the size of a largest set $\chi(t)$ minus 1 among all nodes t of T. A tree decomposition of smallest width is *optimal*. The *treewidth* of a graph G is the width of an optimal tree decomposition of G.

We are going to show Theorem 3 which states that the problems SUBJECTIVE/OBJECTIVE ACCEPTANCE can be decided in linear time for VAFs of bounded treewidth. The proof of this theorem requires some preparation. Let S denote a finite relational structure and φ a sentence in monadic second-order logic (MSO logic) on S. That is, φ may contain quantification over atoms (elements of the universe) of S and over sets of atoms of S. Furthermore, we associate with the structure S its *Gaifman graph* $G(S)$, whose vertices are the atoms of S, and where two distinct vertices are joined by an edge if and only if they occur together in some tuple of a relation of S. The treewidth of S is the treewidth of its Gaifman graph $G(S)$.

We shall use Courcelle's celebrated result [6] that for a fixed MSO sentence φ and a fixed integer w, one can check in linear time whether φ holds for a graph (or more generally, for a relational structure) of treewidth at most w. We use Courcelle's result as laid out in Flum and Grohe's book [12].

First we explain how we represent an instance (F, x_1) of SUBJECTIVE/OBJECTIVE ACCEPTANCE as a relational structure S_F. Let $F = (X, A, V, \eta)$ be a VAF, $x_1 \in X$, and $<$ an arbitrary but fixed linear ordering of V. For every pair of values (u, v) such that $u < v$ and A contains an arc (x, x') with $\eta(x) = u$ and $\eta(x') = v$ or $\eta(x) = v$ and $\eta(x') = u$, we take a new atom $w_{(u,v)}$; let $R_<$ be the set of all such atoms. The universe of S_F is the set $X \cup V \cup R_<$. Furthermore, S_F has one unary relation U_a^* and four binary relations $H_{R_<}$, $T_{R_<}$, B_a and B_η that are defined as follows:

1. $\mathrm{U}_a^*(x)$ if and only if $x = x_1$ (used to "mark" the initial argument).
2. $\mathrm{T}_{R_<}(t, w_{(u,v)})$ if and only if $t = u$ (used to represent the tail relation for $R_<$)
3. $\mathrm{H}_{R_<}(h, w_{(u,v)})$ if and only if $h = v$ (used to represent the head relation for $R_<$)
4. $\mathrm{B}_a(x, y)$ if and only if $(x, y) \in A$ (used to represent the attack relation).
5. $\mathrm{B}_\eta(x, v)$ if and only if $\eta(x) = v$ (used to represent the mapping η).

We shall define two MSO sentences φ_s and φ_o such that φ_s is true for S_F if and only if x_1 is subjectively accepted in F, and φ_o is true for S_F if and only if x_1 is objectively accepted in F.

Before doing so, we establish that the treewidth of S_F is bounded in terms of the treewidth of F. Note that the Gaifman graph for S_F is the graph $G(S_F) = (V_{S_F}, E_{S_F})$ with $V_{S_F} = X \cup V \cup R_<$ and $E_{S_F} = \{\, \{u, v\} \mid (u, v) \in T_{R_<} \cup H_{R_<} \cup B_a \cup B_\eta \,\}$.

Lemma 6. *The treewidth of S_F is at most twice the treewidth of F plus* 1.

Proof. Let $G'(S_F)$ be the graph obtained from $G(S_F)$ after replacing every path of the form $(t, w_{(t,h)}, h)$ by an edge $\{t, h\}$; i.e., $G'(S_F) = (X \cup V, (E_{S_F} \cap \{\, \{u, v\} \mid u, v \in (X \cup V) \,\}) \cup \{\, \{t, h\} \mid (t, h) \in R_< \,\}$. Conversely one can obtain $G(S_F)$ from $G'(S_F)$ by sub-dividing all edges of the form $\{t, h\}$ with a vertex $w_{(t,h)}$. However, subdividing edges does not change the treewidth of a graph [5], hence it suffices to show that the treewidth of $G'(S_F)$ is at most twice the treewidth of F plus 1. Let $\mathcal{T} = (T, \chi)$ be a tree decomposition of the extended graph structure of F. We observe that $\mathcal{T}' = (T, \chi')$ where $\chi'(t) = \chi(t) \cup \{\, \eta(v) \mid v \in \chi(t) \,\}$ is a tree decomposition for $G'(S_F)$ where $|\chi'(t)| \leq 2 \cdot |\chi(t)|$ for all nodes t of T; hence the width of $\mathcal{T}'$ is at most twice the width of $\mathcal{T}$ plus 1. □

For our subsequent considerations it is convenient to introduce the following concepts and notation. Let $D^< = (V, E^<)$ be the directed graph where V is the set of values of F and $E^< := \{\, (u, v) \mid w_{(u,v)} \in R_< \,\}$. Furthermore, for a subset $Q \subseteq E^<$ let $D_Q^< = (V, E_Q^<)$ be the directed graph obtained from $D^<$ by reversing all arcs in Q, i.e., $E_Q^< := \{\, (u, v) \mid (u, v) \in E^< \setminus Q) \,\} \cup \{\, (v, u) \mid (u, v) \in E^< \cap Q \,\}$. We also define the AF $F_Q^< := (X, A_Q^<)$ as the AF obtained from F such that $A_Q^< := \{\, (u, v) \in A \mid (\eta(u), \eta(v)) \notin E_Q^<) \,\}$.

Every audience $\leq$ can be represented by some subset $Q \subseteq E^<$ for which the directed graph $D_Q^<$ is acyclic, and conversely every set $Q \subseteq E^<$ such that $D_Q^<$ is acyclic represents a specific audience $\leq$. This is made precise in the following lemma whose easy proof is omitted due to space limitations.

Lemma 7. *An argument x_1 is subjectively accepted in F if and only if there exists a set $Q \subseteq E^<$ such that $D_Q^<$ is acyclic and x_1 is in the unique preferred extension of $F_Q^<$. An argument x_1 is objectively accepted in F if and only if for every set $Q \subseteq E^<$ such that $D_Q^<$ is acyclic it holds that x_1 is in the unique preferred extension of $F_Q^<$.*

We are now in the position to state the main lemma of this section.

Lemma 8. *There exists an MSO sentence φ_s such that φ_s is true for S_F if and only if x_1 is subjectively accepted in F. Similarly, there exists an MSO sentence φ_o such that φ_o is true for S_F if and only if x_1 is objectively accepted in F.*

Proof. In order to define φ_s and φ_o we need the following auxiliary formulas:
The formula $\text{TH}(t, h, a)$ holds if and only if t is the tail and h is the head of $a \in R_<$:

$$\text{TH}(t, h, a) := \text{T}_{R_<}(t, a) \wedge \text{H}_{R_<}(h, a)$$

The formula $\text{E}(t, h, Q)$ holds if and only if the arc (t, h) is contained in $E_Q^<$:

$$\text{E}(t, h, Q) := \exists a\ [(\neg Qa \wedge \text{TH}(t, h, a)) \vee (Qa \wedge \text{TH}(h, t, a))]$$

The formula $\text{ACYC}(Q)$ checks whether $D_Q^<$ is acyclic[1]:

$$\text{ACYC}(Q) := \neg\exists C\ (\exists x Cx \wedge \forall t \exists h[Ct \rightarrow (Ch \wedge \text{E}(t, h, Q))])$$

The formula $\text{B}'_a(t, h, Q)$ holds if and only if t attacks h in $F_Q^<$:

$$\text{B}'_a(t, h, Q) := \text{B}_a(t, h) \wedge \exists v_h \exists v_t\ [\text{B}_\eta(t, v_t) \wedge \text{B}_\eta(h, v_h) \wedge \neg\text{E}(v_h, v_t, Q)]$$

The formula $\text{ADM}(S, Q)$ checks whether a set $S \subseteq X$ is admissible in $F_Q^<$:

$$\text{ADM}(S, Q) := \forall x \forall y\ [(\text{B}'_a(x, y, Q) \wedge Sy) \rightarrow (\neg Sx \wedge \exists z(Sz \wedge \text{B}'_a(z, x, Q)))]$$

Now φ_s can be defined as follows:

$$\varphi_s := \exists Q\ [\text{ACYC}(Q) \wedge (\exists S(\forall x(\text{U}_a^*(x) \rightarrow Sx) \wedge \text{ADM}(S, Q)))]$$

It follows from Lemma 7 that φ_s is true for S_F if and only if x_1 is subjectively accepted in F. A trivial modification of φ_s gives us the desired sentence φ_o as follows:

$$\varphi_o := \forall Q\ [\text{ACYC}(Q) \rightarrow (\exists S(\forall x(\text{U}_a^*(x) \rightarrow Sx) \wedge \text{ADM}(S, Q)))]$$

It follows from Lemma 7 that φ_o is true for S_F if and only if x_1 is objectively accepted in F and the result follows. □

Proof of Theorem 3. In view of Lemmas 6 and 8 the result now follows by Courcelle's Theorem. □

7 Conclusion

We have studied the computational complexity of persuasive argumentation for value-based argumentation frameworks under structural restrictions. We have established the intractability of deciding subjective or objective acceptance for VAFs with value-width 2 and attack-width 1, disproving conjectures stated by Dunne. It might be interesting to note that our reductions show that

[1]We use the well-known fact that a directed graph contains a directed cycle if and only if there is a nonempty set C of vertices each having an out-neighbor in C.

intractability even holds if the attack relation of the VAF under consideration forms a directed acyclic graph. On the positive side we have shown that VAFs with value-width 2 whose graph structure is bipartite are solvable in polynomial time. These results establish a sharp boundary between tractability and intractability of persuasive argumentation for VAFs with value-width 2. Furthermore we have introduced the notion of the *extended graph structure* of a VAF and have shown that subjective and objective acceptance can be decided in linear-time if the treewidth of the extended graph structure is bounded (that is, the problems are *fixed-parameter tractable* when parameterized by the treewidth of the extended graph structure). This result suggests that the extended graph structure is indeed an appropriate graphical model for studying the computational complexity of persuasive argumentation. It might be interesting for future work to extend this study to other graph-theoretic properties or parameters of the extended graph structure.

References

[1] P. Baroni and M. Giacomin. Semantics of abstract argument systems. In I. Rahwan and G. Simari, editors, *Argumentation in Artificial Intelligence*, pages 25–44. Springer Verlag, 2009.

[2] T. J. M. Bench-Capon. Persuasion in practical argument using value-based argumentation frameworks. *J. Logic Comput.*, 13(3):429–448, 2003.

[3] T. J. M. Bench-Capon, S. Doutre, and P. E. Dunne. Audiences in argumentation frameworks. *Artificial Intelligence*, 171(1):42–71, 2007.

[4] T. J. M. Bench-Capon and P. E. Dunne. Argumentation in artificial intelligence. *Artificial Intelligence*, 171(10-15):619–641, 2007.

[5] H. L. Bodlaender. A tourist guide through treewidth. *Acta Cybernetica*, 11:1–21, 1993.

[6] B. Courcelle. Recognizability and second-order definability for sets of finite graphs. Technical Report I-8634, Université de Bordeaux, 1987.

[7] R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Monographs in Computer Science. Springer Verlag, 1999.

[8] P. M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77(2):321–357, 1995.

[9] P. E. Dunne. Computational properties of argument systems satisfying graph-theoretic constraints. *Artificial Intelligence*, 171(10-15):701–729, 2007.

[10] P. E. Dunne and T. J. M. Bench-Capon. Complexity in value-based argument systems. In J. J. Alferes and J. A. Leite, editors, *Logics in Artificial Intelligence, 9th European Conference, JELIA 2004, Lisbon, Portugal, September 27-30, 2004, Proceedings*, volume 3229 of *Lecture Notes in Computer Science*, pages 360–371. Springer Verlag, 2004.

[11] P. E. Dunne and M. Wooldridge. Complexity of abstract argumentation. In L. Rahwan and G. R. Simari, editors, *Argumentation in Artificial Intelligence*, pages 85–104. Springer Verlag, 2009.

[12] J. Flum and M. Grohe. *Parameterized Complexity Theory*, volume XIV of *Texts in Theoretical Computer Science. An EATCS Series*. Springer Verlag, 2006.